Minimal non-extensible precolorings and implicit-relations

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Abstract. In this paper I study a variant of the general vertex coloring problem called precoloring. Specifically, I study graph precolorings, by developing new theory, for characterizing the minimal non-extensible precolorings. It is interesting per se that, for graphs of arbitrarily large chromatic number, the minimal number of colored vertices, in a non-extensible precoloring, remains constant; only two vertices u, v suffice. Here, the relation between such u, v is called an implicit-relation, distinguishing two cases: (i) implicit-edges where u, v are precolored with the same color and (ii) implicit-identities where u, v are precolored distinct.

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1. Introduction

In this work, a formal definition of a theory of graph-chromatic implicit-relations is presented. This theory can be described a specialization of a variant of the general vertex coloring problem called precoloring, specifically, for characterizing minimal non-extensible precolorings. For a good reference on graph coloring (a.k.a. chromatic graph theory) the reader can see the introductory book of Chartrand and Zhang [2], and also the book of Jensen and Toft [4] with interesting open problems. Chartrand and Zhang [2, pp. 240] open the precoloring section affirming that:

> ...What we are primarily interested in, however, is whether a k-precoloring of G, where $k \geq \chi(G)$, can be extended to a k-coloring of G...

However, in this paper, I show that we should pay attention, not only to $(k \geq \chi(G))$-precolorings of G, but, just to the 2-precolorings of G, since this are the minimal non-extensible precolorings, without regard of $\chi(G)$. This

work is also relevant in connection with all theorems and problems exposed by Chartrand and Zhang, in the section about "Precoloring Extensions of Graphs" [2, Chap.9, Sec.3].

It is easy to show that for an arbitrarily large chromatic number k, the minimal number of colored vertices, in a non-extensible precoloring, for some k-chromatic graph G, remains constant since only two vertices u, v suffice (however it remains open and, up to my knowledge, there is no study about the minimal number of vertices of a non-extensible precoloring for some given particular graph).

We call the relation between such u, v an implicit-relation distinguishing two cases:

(i). implicit-edges, where u, v has the same color, and this is a minimal non-extensible precoloring.

(ii). implicit-identities, where u, v has distinct color, and this is a minimal non-extensible precoloring.

Additionally, the notion of implicit-edge can be derived from the observation that, when finding a k-coloring of some k-chromatic graph G, there may be some independent set $S \subset V(G)$ such that there is no k-coloring of G where all the vertices in S receive the same color. Thus, when $|S| = 2$, we say that the 2-element subset $S = \{u, v\}$ is an implicit-edge, based on the intuition that the k-colorings of G are the same as if $uv \in E(G)$. Conversely, implicit-identities are, in most cases, the opposite-equivalent of implicit-edges. It is a relation in which vertices must receive always the same color being intuitively the same as if $G = G - uv/u, v$ (the vertex identification of u and v).

Hence, given a k-chromatic graph G:

(iii). there is an implicit-edge when u, v receive *different* colors in every k-coloring of G.

(iv). there is an implicit-identity when u, v receive *the same* color in every k-coloring of G.

Implicit-relations have direct application, for instance, in the study of two very important open problems in graph theory, namely: the double-critical graphs problem and the Hadwiger's conjecture.

In short, a way to express the Hadwiger's conjecture is the following: every k-chromatic graph can be contracted (by successive edge contractions or edge and vertex deletions) into a complete graph on k vertices (K_k). An equivalent formulation is to show that there are no contraction-critical graphs (i.e. a graph where each minor has lower chromatic number) different from K_k. The relation with implicit-edges is clear since contraction critical graphs cannot have minors with implicit-edges in the edge set. Hence a deep study of this concept is a way to approach the conjecture.

In this paper, in order to study and characterize such concepts, a series of theorems are presented.

2. Preliminary definitions and basic terminology

Unless we state it otherwise, all graphs in this work are connected and simple (finite, and have no loops or parallel edges).

Partitioning the set of vertices $V(G)$ of a graph G into separate classes, in such a way that no two adjacent vertices are grouped into the same class, is called the vertex graph coloring problem. In order to distinguish such classes, a set of colors C is used, and the division into these *(color) classes* is given by a proper-coloring (we will use here just the single term coloring) $c : V(G) \to \{1...k\}$, where $c(u) \neq c(v)$ for all $uv's$ belonging to the set of edges $E(G)$ of G. A *k-coloring* of G is a coloring that uses exactly k colors. The *Chromatic number* of a graph $\chi(G)$ is the minimum number such that there is a $\chi(G)$-coloring of a graph G. Thus, if $\chi(G) \leq k$ then one says that G is *k-colorable* (i.e. G can be colored with k different colors) and if $\chi(G) = k$ then one says that G is *k-chromatic*.

A *precoloring* of a graph G, is a coloring $p : W \to \{1...k\}$ of $W \subset V(G)$ such that $p(u) \neq p(v)$ if $u, v \in W$ and $uv \in E(G)$. A precoloring p of G can be *extended* to a coloring of all the vertices of G when there is at least one coloring $c : V(G) \to \{1...k\}$ of G such that $c(u) = p(u)$ for all $u \in W$, otherwise it is said that p is non-extensible. A *k-precoloring* p is a precoloring of G such that p uses only k colors.

An *independent set* (also called *stable set*) $I = \{u, v, w, ...\}$ is a set of vertices of a graph G such that there are no edges between any two vertices in I, i.e, if $\{u, v\} \in I$ then $uv \notin E(G)$. The set $I(G)$ will denote the set of all independent sets of graph G.

The set off all adjacent vertices to a vertex $u \in V(G)$ is called its *neighborhood* and is denoted by $N_G(u)$ (when it is clear to which graph we are referring to, we will use simply $N(u)$, i.e. omitting the graph). The *closed neighborhood* of a vertex u, denoted by $N[u]$, includes also the vertex u, i.e. $N(u) \cup \{u\}$.

The *degree* of a vertex u, $deg(u)$, is equal to the cardinality of its neighborhood $deg(u) = |N(u)|$. A *complete vertex* is any $u \in V(G)$ such that $N[u] = V(G)$ and a graph is called a *complete graph* if every vertex is a complete vertex.

A *path* P_n, of order n and length ℓ, is a sequence of n joined vertices such that the travel from the *start* vertex to the *end* vertex passes trough ℓ edges. A *simple path* has no repeated vertices.

Vertex deletions and additions are denoted as $G - u$ and $G + u$ respectively, for a graph G and a vertex u. Edge deletions and additions are denoted as $G - uv$ and $G + uv$, or simply $G - e$ or $G + e$ respectively, for a graph G and edge $e = uv$.

A *vertex identification*, denoted by $G - uv/u, v$, is the process of replacing two *non-adjacent* vertices u, v of a graph G, i.e $uv \notin E(G)$, by a new vertex w such that $N(w) = N(u) \cup N(v)$.

An *edge contraction*, denoted by G/uv or G/e, is the process of replacing two *adjacent* vertices u, v of G, i.e $uv \in E(G)$, by a new vertex w such that $N(w) = N(u) \cup N(v)$.

A *vertex contraction*, denoted by $G/u, v$, is the process of replacing two vertices u, v of a graph G, by a new vertex w such that $N(w) = N(u) \cup N(v)$. Hence *vertex contractions* include both cases: vertex identifications and edge contractions.

A graph H is called a *minor* of the graph G, denoted as $H \prec G$, if H is isomorphic to a graph that can be obtained from a subgraph of G by zero or more edge deletions, edge contractions or vertex deletions on a subgraph of G. In particular, G is minor of itself.

An element x of a graph G is called *critical* if $\chi(G - x) < \chi(G)$. If all the vertices of a graph G are critical we say that G is *vertex-critical* and if every element (vertex or edge) of G is critical we say that G is a *critical graph* and more specifically if $\chi(G) = k$ we say that G is k-critical.

An *edge-subdivision* of an edge e of a graph G is the subdivision of some $e \in E(G)$ with endpoints $\{u, v\}$ resulting in a graph containing one new vertex w, and with an edge set replacing e by two new edges uw and wv i.e., $H = G - e + w + uw + wv$.

Given a (pre)coloring $c : V(G) \to \{1...k\}$ of a graph G, a *2-color-chain* Ω_{uv} (a.k.a *Kempe's chain*) is a simple maximal bipartite connected subgraph $B \subset G$ such that $\{u, v\} \in V(B)$ and every vertex in B has either the color $c(u)$ or $c(v)$. *Flipping* a chain Ω_{uv} is the process of swapping the color of vertices with color $c(u)$ to $c(v)$ and, respectively, vertices with color $c(v)$ to $c(u)$.

3. Implicit relations

The chromatic implicit relations are mainly defined by their two most basics concepts, implicit-edges and implicit-identities.

The notion of implicit-edge is based on the observation that, when finding a k-coloring of some k-chromatic graph G, there may be some independent set $S \in I(G)$ such that there is no k-coloring of G where all the vertices in S receive the same color. Thus, when $|S| = 2$ we say that the 2-elements set $S = \{u, v\}$ is an implicit-edge. Respectively, implicit-identities are, in most cases, the opposite-equivalent of implicit-edges. While implicit-edges are defined as a relation in which vertices receive always different color implicit-identities are the opposite relation, that is, the vertices receive always the same color.

Definition 3.1. Given a k-chromatic graph G, we say that $\{u, v\} \in V(G)$ is an *implicit-edge* **iff** the set of all k-colorings of $G - uv$ where u and v receive the same color is the empty set:

$$\{c \in \Phi(G - uv) \mid c(u) = c(v)\} = \emptyset, \tag{3.1}$$

where $\Phi(G - uv)$ is the set of all k-colorings of $G - uv$ and $c(u), c(v)$ are the colors of vertices u, v respectively, as assigned by a particular coloring (c) of G.

Remark 3.2. Also, we must note that an implicit-edge could belong to the set of edges of G or not, i.e. $\{u, v\}$ is also an implicit-edge in the graph $G - ij$. This can also be viewed as a precoloring that can't be extended to all G (i.e. a non-extensible precoloring) by taking any $p : W \to C$ of $W \subset V(G)$ such that $p(u) = p(v)$.

Definition 3.3. Given a k-chromatic graph G, we say that $\{u, v\} \in V(G)$ is an *implicit-identity* **iff** the set of all k-colorings of $G - uv$ where u and v receive different color is the empty set:

$$\{c \in \Phi(G - uv) \mid c(u) \neq c(v)\} = \emptyset, \tag{3.2}$$

where $\Phi(G - uv)$ is the set of all k-colorings of $G - uv$ and $c(u), c(v)$ are the colors of vertices u, v respectively, as assigned by a particular coloring (c) of G.

Remark 3.4. Note that contrary to implicit-edges, if we add the edge $e = uv$ to G the resulting graph $G + e$ will not be k-colorable. This can also be viewed as a 2-precoloring that can't be extended to all G by taking any $p : W \to \{1..k\}$ of $W \subset V(G)$ such that $p(u) \neq p(v)$.

The trivial case of implicit-edges arise in bipartite (2-chromatic) graphs.

Lemma 3.5. Given a bipartite graph G, $\{u, v\} \in V(G)$ is an implicit-edge **iff** there is an odd-path P_{2n} starting at u and ending at v. (without provided proof)

Let us take the path P_4, as an illustrative example:

$$P_4 = \ u\circ\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\bullet v, \tag{3.3}$$

where $\{u, v\}$ is an implicit-edge of P_4. If we identify vertices $\{u, v\}$ we get the K_3 graph. Also if we add the edge uv to P_4 we get a square C_4 and then all edges become implicit-edges.

More complex examples are shown in Fig. 1. for 3-chromatic planar graphs. Vertices $\{u, v\}$ forms an implicit edge since a 1-precoloring p of $\{u, v\}$, i.e. $p(u) = p(v)$, can't be extended to a 3-coloring of G.

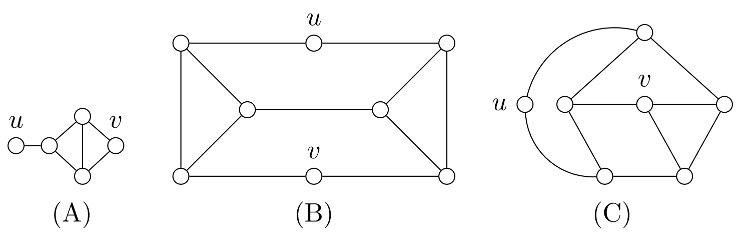


FIGURE 1. Implicit edges (u, v) in 3-chromatic planar graphs.

Implicit-identities in bipartite graphs are defined very easy

Lemma 3.6. Given a bipartite graph G, $\{u, v\} \in V(G)$ is an implicit-identity **iff** there is an even-path P_{2n+1} starting at u and ending at v. (without provided proof)

Let us take the path P_5, as an illustrative example:

$$P_5 = \ u\circ\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\circ v, \tag{3.4}$$

where $\{u, v\}$ is an implicit-identity of P_5. If we add the edge uv then we obtain a 3-chromatic graph, a 5-cycle C_5.

Theorem 3.7. Given a k-chromatic graph G, $\{u, v\} \in V(G)$ is:

1. an implicit-edge **iff** there is no independent set $S \in I(G-uv)$ such that $\{u, v\} \in S$ and $\chi(G - S) < k$.
2. an implicit-identity **iff** there is no independent set $S \in I(G - u)$ such that $v \in S$ and $\chi(G - S) < k$.

$$(1)\ \{S \in I(G-uv) \mid \{u, v\} \in S,\ \chi(G-S) < \chi(G)\} \qquad = \emptyset \tag{3.5}$$

$$(2)\ \{S \in I(G-u) \mid v \in S,\ \chi(G-S) < \chi(G)\} \qquad = \emptyset \tag{3.6}$$

Proof.

1. proof:
   (a) Assume that $\{u, v\}$ is an implicit-edge, but suppose that there is an independent set $S \in I(G - uv)$ where $\{u, v\} \in S$ such that $\chi(G - S) < k$. Then, we can find a (k-1)-coloring of $G - uv - S$ and restore all the vertices in S with color k. Hence there will be a k-coloring of $G - uv$ where $c(u) = c(v)$, which is a contradiction.
   (b) Assume that $\{u, v\}$ is **not** an implicit-edge. Then, by definition, there is a k-coloring of $G - uv$ such that $c(u) = c(v)$. Hence, there will be an independent set $S \in I(G - uv)$ where $\{u, v\} \in S$ such that $\chi(G - S) < k$.
2. proof:
   (a) Assume that $\{u, v\}$ is an implicit-identity, but suppose that there is an independent set $S \in I(G-u)$ such that $v \in S$ and $\chi(G-S) < k$. Then, we can find a (k-1)-coloring of $G - S$ and restore all the vertices in S with color k. Hence there will be a k-coloring of G where $c(u) \neq (v)$ since $c(v) = k$, which is a contradiction.
   (b) Assume that $\{u, v\}$ is **not** an implicit-identity. Then, by definition, there is a k-coloring of G such that $c(u) \neq c(v)$. Hence, there will be an independent set $S \in I(G-u)$ such that $v \in S$ and $\chi(G-S) < k$.

□

Definition 3.8. Given a k-chromatic graph G, we say that an independent set $S \subset V(G)$ is critical **iff** $\chi(G - S) = k - 1$ (i.e. S is a color class of some k-coloring of G)

From this definition it follows immediately the that critical independent sets does not contain implicit-edges, due to theorem 3.7. Hence every maximal independent set (also a minimal dominating set) containing an implicit-edge or containing exactly one vertex of an implicit-identity is not critical.

Theorem 3.9. If there is an implicit-relation (edge or identity) $\{u, v\}$ in a k-chromatic graph G and $H = G - S$ is the resulting (k-1)-chromatic graph obtained by removing a critical independent set S from G then $\{u, v\} \in V(H)$ has the same implicit-relation (edge or identity, respectively) in H.

Proof.

1. Let us suppose that $\{u, v\}$ is **not** an implicit-edge of H. Now, find a (k-1)-coloring of H such that $c(u) = c(v)$ and then restore the critical independent set S with all vertices colored the same. Thus, the graph $G = H + S$ has a k-coloring where $c(u) = c(v)$ which is a contradiction.
2. Let us suppose that $\{u, v\}$ is **not** an implicit-identity of H. Now, find a (k-1)-coloring of H such that $c(u) \neq c(v)$ and then restore the critical independent set S with all vertices colored the same. Thus, the graph $G = H + S$ has a k-coloring where $c(u) \neq (v)$ which is a contradiction.

□

Theorem 3.10. If there is an implicit-relation (edge or identity) $\{u, v\}$ in a k-chromatic graph G then $\{u, v\} \in V(H)$ has the same implicit-relation (edge or identity, respectively) in H, for all subgraphs $H \subset G$ such that:

$$H = G - \{S_1, \dots, S_\ell\} \text{ where } 0 \leq \ell \leq k - 2,$$

where each S_i a critical independent set not containing u nor v.

Proof.
By reiterative application of theorem 3.9 we get an inductive proof with base case ℓ=0 and theorem 3.9 for $\ell + 1$. □

Theorem 3.11. If G is a k-chromatic graph with an implicit-relation $\{u, v\} \in V(G)$:

1. if $\{u, v\}$ is an implicit-edge then every k-coloring of G contains at least one chain Ω_{uv} with colors $c(u), c(v)$ where $\{u, v\} \in \Omega_{uv}$
2. if $\{u, v\}$ is an implicit-identity then every k-coloring of G contains at least (k-1)-chains Ω_{ui} with colors $c(u), c(i)$ where $\{u, v\} \in \Omega_{ui}$ $\forall i =$ 1...$(k - 1)$.

Proof.

1. Otherwise we can simply flip the color of vertex u with other vertex w of color $c(v)$ obtaining a k-coloring of $G - uv$ where $c(u) = c(v)$ which is a contradiction since $\{u, v\}$ is an implicit-edge.

2. Otherwise we can simply flip the color of vertex u with some adjacent vertex i and, since there is no Ω_{ui} chain, there is a k-coloring of G where $c(u) \neq c(v)$ but $\{u, v\}$ is an implicit-identity which is a contradiction.

□

4. Chromatic Polynomials

The chromatic polynomial $P(G, k)$ for a given graph G is a polynomial which encodes the number of different k-colorings of G, we can denote the number of k-colorings of a graph G as $P(G, k)$. Chromatic Polynomials satisfies the next two relations [3](pp.4–6):

$$P(G, k) = P(G - e, k) - P(G/e, k) \tag{4.1}$$

$$P(G, k) = P(G + e, k) + P(G/e, k) \tag{4.2}$$

A direct consequence of (4.1), (4.2) and the definition of implicit-relations is presented in two new theorems:

Theorem 4.1. If G is a k-chromatic graph and $\{u, v\}$ is an implicit-edge of G then $P(G/x, y) = 0$, that is, the graph $G/x, y$ is not k-colorable.

Proof.
If $e = \{u, v\}$ an implicit-edge of G the it follows that $P(G, k) = P(G - e, k)$ or $P(G, k) = P(G + e, k)$, hence:

$$P(G, k) = P(G - e, k) \Longrightarrow P(G/e, k) = 0 \text{ due to (4.1).} \tag{4.3}$$

$$P(G, k) = P(G + e, k) \Longrightarrow P(G/e, k) = 0 \text{ due to (4.2).} \tag{4.4}$$

□

Corollary 4.2. Given a minor-closed class $\mathcal{G}$ of graphs (i.e. if $G \in \mathcal{G}$ and $G \geq H$, then $H \in \mathcal{G}$). If G is a k-chromatic graph in $\mathcal{G}$ and $\{e\}$ a drawn-implicit-edge then G/e is a $(k+1)$-chromatic graph in $\mathcal{G}$.

Theorem 4.3. If G is a k-chromatic graph and $\{u, v\}$ is an implicit-identity of G then $P(G, k) = P((G - uv/u, v), k)$.

Proof.
Being $\{u, v\}$ an implicit-identity of a k-chromatic graph G it follows that $P(G + e, k) = 0$ hence:

$$P(G + e, k) = 0 \Longrightarrow P(G, k) = P((G - uv/u, v), k) \text{ due to (4.2).} \tag{4.5}$$

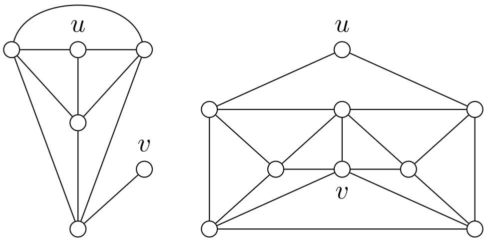
□

5. Planar graphs

A graph is called planar if it can be drawn in a plane without edge crossings. Planar graphs poses the next important properties:

- Every planar graph is four colorable [1, 5].
- Planar graphs are closed under edge contraction, that is, every edge contraction of a planar graph results in a new planar graph.

So, are there implicit-relations in 4-chromatic planar graphs? The question is affirmative, there are implicit-relations in 4-chromatic planar graphs, as can be seen in the graphs of Fig. 2.



(A) (B)

FIGURE 2. Implicit edges (u, v) in 4-chromatic planar graphs.

The main observation from Fig. 2. is that the implicit-edge can not be drawn without edge line crossings. It is straightforward to proof that if implicit-edges in 4-chromatic planar graphs could be drawn without edge line crossings then this would be a counter example to the four-color theorem due to the corollary of Theorem 4.1. Even more, the problem of determining if there are no implicit-edges in 4-chromatic planar graphs without edge-lines crossings is equivalent to a proof of the four-color theorem and conversely:

Theorem 5.1. There is a 5-chromatic planar graph **iff** exist a 4-chromatic planar graph G such that $u, v \in V(G)$ is an implicit-edge and $G/u, v$ is planar.

Proof.

1. Let G be a critical 5-chromatic planar graph. Then by doing an edge-subdivision $H = G - e + w + uw + wv$ the resulting edges uw and wv of the 4-chromatic planar graph H are implicit-edges and have no edge-line crossings.
2. The backward implication is immediate from theorem 4.1.

□

Finally, can it be proved, without using the 4-colors theorem, that:

Theorem 5.2. If G is a 4-chromatic planar graph such that $u, v \in V(G)$ is an implicit-relation (edge or identity) then $G + uv$ is not a planar graph.

6. Critical graphs

Theorem 6.1. If G is a k-chromatic graph such that $w \in V(G)$ is critical then:

1. if $\{u, v\}$ is an implicit-identity then $uw \in E(G)$ and $vw \in E(G)$.
2. if $\{u, v\}$ is an implicit-edge then at least $uw \in E(G)$.

Proof.

1. Otherwise $G - u - w$ is (k-1)-chromatic which also contradicts theorem 3.7.
2. Otherwise $G-u-v-w$ is (k-1)-chromatic which contradicts theorem 3.7.

□

6.1. Double-critical graphs

A k-chromatic graph G is called double-critical if for every edge $uv \in E(G)$ the graph $G-u-v$ is $(k-2)$-colorable. A deep structural result about double-critical graphs can be proved easily with the help of implicit-relations:

Theorem 6.2. If G is a k-chromatic double-critical graph and $uv \in E(G)$ then u and v have at least k-1 common neighbors, i.e.:

$$\chi(G-u-v) = \chi(G)-2 \rightarrow |N(u) \cap N(v)| \geq \chi(G)-2, \text{ for all } uv \in E(G) \quad (6.1)$$

Proof. Since in the (k-1)-chromatic graph $G-uv$, $\{u, v\}$ is an implicit-identity then, by theorem 3.11, there will be $k-2$ color-chains containing $\{u, v\}$. Now since, G is double-critical there is a (k-1)-coloring of $G - uv$ such that $\{u, v\}$ are the only vertices having the color k-1. Hence each color-chain can only pass trough a common neighbor, i.e. one neighbor for each color-chain. □

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